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ON THE SOLUTION OF CONICAL SHELLS OF LINEARLY VARYING THICKNESS SUBJECTED TO LATERAL NORMAL LOADS*

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Abstract—The solution of elastic conical shells of linearly varying thickness is hinged on an eighth degree characteristic equation. In this paper, a method of solving this equation is presented which results in a general asymptotic solution. The particular solution due to a lateral normal lqad which is constant along meridians and has a sinusoidal distribution in the circumferential direction is given. Included is also an illustrative example of a semicircular cone which is simply supported along two generators with one end fixed and the other end free.

INTRODUCTION

THE theory of conical shells of linearly varying thickness in the framework of generalized plane stresses of linear theory of elasticity along with a general approach of solving the basic equations has been given in [1]. The three homogeneous equilibrium equations in terms of three displacement components were solved by the classical method of separation of variables; in turn the solutions were hinged on an eighth degree characteristic equation. There are, however, no complete solutions available thus far. This is perhaps due to the high degree of complexity involved.

In this work an attempt is made to get the solution to an extent that it is general and usable for practical design purposes.

By recognizing the significance of a parameter which depends on the ratio of the thickness to length, the characteristic equation given in [1] is presented in a different form. A method which is approximate, but consistent with the theory, is proposed to solve the equation.

An asymptotical solution is obtained. The solution consists of two parts: membrane and bending. These two parts are coupled by the lateral displacements. It is found that the order of magnitude of the displacements, stress resultants, and stress couples agrees with that indicated in a general discussion by Steele [2].

Generally one would expect no difficulties in obtaining the particular solutions of the system due to a lateral normal load. When the load is uniformly distributed along meridians, the solution, however, is at a singularity of the system in the asymptotical case.† The particular solution of such a case is given, including a numerical example.

The basic equations of the system are essentially given in [1]. For completeness and comprehension, nevertheless, most equations will be symbolically repeated.

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† Near the apex of a cone, there is another kind of singularity. See [3].

BASIC AND CHARACTERISTIC EQUATIONS

Let θ, s be circumferential and meridional coordinates of the middle surface of an isotropic conical cone and u, v, w be circumferential, meriodional, and normal displacement components, respectively. Outward w is positive. When the thickness of shell h is proportional to s and independent of θ, one has

$$h = \delta s \tag{1}$$

where δ is a constant which for thin shells is very small. The elastic law assumes relationships between the stress resultants and displacement components in the following forms:

$$N_s, N_\theta, N_{s\theta}, N_{\theta s} = \frac{E\delta}{1-\nu^2}[N_1(u, v, w)+kN_2(u, v, w)]$$

$$M_s, M_\theta, M_{s\theta}, M_{\theta s} = \frac{E\delta}{1-\nu^2}k[M(u, v, w)] \tag{2}$$

in which $N_s, \ldots, M_{\theta s}$ are stress resultants and stress couples per unit length; N_1, N_2, and M are functions of the displacements and their space derivatives; E is Young's modulus of elasticity; ν is Poisson's ratio; and

$$k = \frac{\delta^2}{12} \tag{3}$$

The elastic law (2) may be regarded as the result of series expansions of the stresses and displacements in the parameter k with only the terms of zero and first order of k retained.

By dropping the identical one, the other five equations of equilibrium are in a form

$$F(N_s, \ldots, M_{\theta s}, Q_s, Q_\theta, P_r, P_s, P_\theta) = 0 \tag{4}$$

where Q_s and Q_θ are the transverse shear forces per unit length and P_r, P_s, and P_θ are surface loads per unit area in their respective directions.

When the lateral normal load P_r only is considered* and Q_s and Q_θ are eliminated from (4), three equations result as follows:

$$f_l(N_s, \ldots, M_{\theta s}) = P_r s^2 \delta_{lr} \tag{5}$$

in which δ_{lr} is the Kronecker delta and the subscript l represents θ, s, and r.

Substitution of elastic law (2) for equations (5) results in three equations of equilibrium in terms of the three displacements:

$$f_l(u, v, w) = sP_r\frac{1-\nu^2}{E\delta}\delta_{lr}. \tag{6}$$

Consider a segment of cone being bounded by $\theta = 0$ and $\theta_1(\leq 2\pi)$ and $s = L_1$ and L, $L_1 < L$. Introducing a nondimensional variable

$$y = \sqrt{\left(\frac{s}{L}\right)} \tag{7}$$

* When the other load exists, one may follow a similar procedure and by superposition get the appropriate solution.

the displacement functions may now be assumed in the forms:

$$u = A_n y^{\lambda_n - 1} \frac{\sin}{\cos} \frac{n\pi\theta}{\theta_1}$$

$$v = B_n y^{\lambda_n - 1} \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1}$$

$$w = C_n y^{\lambda_n - 1} \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1} \tag{8}$$

in which A_n, B_n, C_n and λ_n are constants to be determined. Physically speaking, the upper set of the sinusoidal functions in (8) is for a complete cone while the lower one is for a segment of cone ($\theta_1 < 2\pi$) with two generator edges simply supported so that along $\theta = 0$ and θ_1

$$w = 0, \qquad v = 0, \qquad N_\theta = 0 \qquad \text{and} \qquad M_\theta = 0. \tag{9}$$

The reactions along the two generator edges are given by

$$S_\theta = Q_\theta + \frac{\partial M_{\theta s}}{\partial s} \qquad \text{at } \theta = 0 \text{ and } \theta_1 \tag{10}$$

where S_θ is transverse shearing force at a section perpendicular to the θ direction. The shearing force Q_θ may be obtained from equation (4).

Let

$$P_r = p_{rn}(y) \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1}. \tag{11}$$

Substitution of this load function and assumed displacements (8) into equations (6) yields the following three equations:

$$\begin{aligned} d_{11}A_n + d_{12}B_n + d_{13}C_n &= 0 \\ d_{21}A_n + d_{22}B_n + d_{23}C_n &= 0 \\ d_{31}A_n + d_{32}B_n + d_{33}C_n &= Lp_{rn}(y)y^{3-\lambda_n}\frac{1-\nu^2}{E\delta} \end{aligned} \tag{12}$$

where d_{ij} (functions of λ, material, and geometrical constants) are given in [1], (p. 400) except signs of ($\pm$) shall be added to d_{12} and d_{13}. These plus and minus signs correspond to the upper and lower set of sinusoidal functions henceforth.

In order to have non-trivial homogeneous solutions of the system of equations (12), the determinant of the coefficients must vanish. This results in an eighth degree characteristic equation for λ_n. Neglecting the terms of second and higher powers of k as it has been done in the derivation of elastic law (2), the characteristic equation is obtained in the following form:

$$G[\lambda_n^4 - 10\lambda_n^2 + 9] + k[\lambda_n^8 - g_6\lambda_n^6 + g_4\lambda_n^4 - g_2\lambda_n^2 + g_0] = 0 \tag{13}$$

in which

$$G = 16(1-\nu^2)\tan^2\alpha \tag{14}$$

where α is the angle between a normal to the middle surface and the axis of the cone. The coefficients $g_6, \ldots, g_0$ in (13) are the same as those given in [1] (p. 401), but omitting the terms with the parameter k. The terms with k have been placed in the first bracket in (13).

In view of the approximation made in the derivation of equation (13), the following approximate method of solution is suggested for this equation. Introducing

$$\lambda_n^2 = X_{n0} + kX_{n1} \tag{15}$$

into equation (13) results in a sequence of equations associated with the various powers of k. The equations associated with the two lowest powers of k are

$$X_{n0}^2 - 10X_{n0} + 9 = 0$$

and

$$X_{n0}^4 - g_6 X_{n0}^3 + g_4 X_{n0}^2 - g_2 X_{n0} + g_0 + 2G(X_{n0} - 5)X_{n1} = 0$$

which give two sets of X_{n0} and X_{n1}. Then equation (15) provides two roots of λ_n^2 which in turn give four roots of λ_n:

$$\lambda_{n\frac{1}{2}} = \pm\left(1 + k\frac{1 - g_6 + g_4 - g_2 + g_0}{8G}\right)^{\frac{1}{2}}$$

$$\lambda_{n\frac{3}{4}} = \pm\left(9 - k\frac{9^4 - 9^3 g_6 + 9^2 g_4 - 9g_2 + g_0}{8G}\right)^{\frac{1}{2}}. \tag{16}$$

Substituting the two determined roots of λ_n^2 denoted by P $(= \lambda_{n\frac{1}{2}}^2)$ and Q $(= \lambda_{n\frac{3}{4}}^2)$ into equation (13) yields a quadratic equation of λ_n^2 which gives the other four roots of λ_n:

$$\lambda_{n\frac{56}{78}} = \pm\left\{\tfrac{1}{2}(g_6 - P - Q) \pm i\left[\frac{1}{PQ}\left(g_0 + \frac{9G}{k}\right) - \tfrac{1}{4}(g_6 - P - Q)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \tag{17}$$

Hence the eight roots of λ_n group into two, four each. One group is of real numbers; the other is of complex numbers.

The next step, as a routine, is to solve for A_n and B_n in terms of C_n for each root of λ_n from any two of the homogeneous equations of equations (12). The eight constants C_n shall be determined by eight conditions at $y = \sqrt{(L_1/L)}$ and 1. The boundary conditions along the generator edges are satisfied by the choice of sinusoidal functions of the angle θ. At the two circular edges one has the following four boundary conditions at each edge. For a built-in edge:

$$u = 0, \quad v = 0, \quad w = 0, \quad \text{and} \quad \partial w/\partial s = 0 \tag{18}$$

and for a free edge:

$$N_s = 0, \quad M_s = 0, \quad S_s = 0, \quad \text{and} \quad T_s = 0 \tag{19}$$

where

$$S_s = Q_s + \frac{1}{s}\frac{\partial M_{s\theta}}{\partial \theta}\sec\alpha$$

$$T_s = N_{s\theta} - \frac{M_{s\theta}}{s}\tan\alpha \tag{20}$$

are the transverse and tangential shearing forces at sections perpendicular to the s-direction, respectively. The shearing force Q_s can be obtained from equations (4). For a simply supported edge:

$$w = 0, \qquad M_s = 0, \qquad N_s = 0, \qquad \text{or} \quad v = 0$$

and

$$T_s = 0 \qquad \text{or} \quad u = 0.$$

ASYMPTOTIC SOLUTIONS

As the parameter k approaches zero, the two groups of roots λ_n reach at the following asymptotic values:

$$\lambda_{\substack{1\\2}} = \pm 1, \qquad \lambda_{\substack{3\\4}} = \pm 3 \tag{21}$$

$$\lambda_{\substack{5\\6}} = \rho(1 \pm i), \qquad \lambda_{\substack{7\\8}} = -\rho(1 \pm i) \tag{22}$$

where

$$\rho \equiv \left| \frac{(\sqrt{2})}{2}\left(\frac{G}{k}\right)^{\frac{1}{4}} \right|. \tag{23}$$

The subscript n has been and henceforth will be dropped for simplicity.

When the first group of λ, λ_i ($i = 1, 2, 3$, and 4) is substituted into the first two equations (12) to eliminate A_i and B_i, and keeping only the leading terms, solutions (8) assume the following forms:

$$\begin{aligned} u^{\mathrm{I}} &= \mp m \tan\alpha \left\{ \frac{C_1}{m^2-1} + \frac{C_2}{m^2-2(1-\nu)}\frac{1}{y^2} + \frac{C_3}{m^2}y^2 + \frac{4+4\nu-m^2}{m^2(7-2\nu-m^2)}\frac{C_4}{y^4} \right\} \frac{\sin}{\cos}\frac{n\pi\theta}{\theta_1} \\ v^{\mathrm{I}} &= \tan\alpha \left\{ \frac{C_1}{m^2-1} + \frac{2C_2}{m^2-2(1-\nu)}\frac{1}{y^2} + \frac{3C_4}{m^2-7+2\nu}\frac{1}{y^4} \right\} \frac{\cos}{\sin}\frac{n\pi\theta}{\theta_1} \\ w^{\mathrm{I}} &= \{C_1 + C_2 y^{-2} + C_3 y^2 + C_4 y^{-4}\} \frac{\cos}{\sin}\frac{n\pi\theta}{\theta_1} \end{aligned} \tag{24}$$

where

$$m = \frac{n\pi}{\theta_1}\sec\alpha.$$

When the second group of λ, λ_j ($j = 5, 6, 7$, and 8) is used following the similar procedure, and using some identities to convert the complex expressions into real, one obtains the following solutions:

$$\begin{aligned} u^{\mathrm{II}} &= \mp 2(2+\nu) m \tan\alpha \frac{1}{\rho^2} y^{-1}\{y^{\rho}[C_6\cos(\rho\ln y) - C_5\sin(\rho\ln y)] - y^{-\rho}[C_8\cos(\rho\ln y) \\ &\quad - C_7\sin(\rho\ln y)]\} \frac{\sin}{\cos}\frac{n\pi\theta}{\theta_1} \\ v^{\mathrm{II}} &= -\nu\tan\alpha\frac{1}{\rho}y^{-1}\{y^{\rho}[(C_5 - C_6)\cos(\rho\ln y) + (C_5 + C_6)\sin(\rho\ln y)] \\ &\quad - y^{-\rho}[(C_7 + C_8)\cos(\rho\ln y) - (C_7 - C_8)\sin(\rho\ln y)]\} \frac{\cos}{\sin}\frac{n\pi\theta}{\theta_1} \\ w^{\mathrm{II}} &= y^{-1}\{y^{\rho}[C_5\cos(\rho\ln y) + C_6\sin(\rho\ln y)] + y^{-\rho}[C_7\cos(\rho\ln y) + C_8\sin(\rho\ln y)]\} \frac{\cos}{\sin}\frac{n\pi\theta}{\theta_1}. \end{aligned} \tag{25}$$

It is noted that the solutions of the first group are simply those of membrane theory.

Based on solutions (24) and (25) one may establish the orders of magnitude of the displacement components†

$$u^{\mathrm{I}}, v^{\mathrm{I}}, w^{\mathrm{I}}, w^{\mathrm{II}} = O\left(\frac{1}{\rho^0}\right)$$

$$v^{\mathrm{II}} = O\left(\frac{1}{\rho}\right) \quad \text{and} \quad u^{\mathrm{II}} = O\left(\frac{1}{\rho^2}\right). \tag{26}$$

Due to u^{I}, v^{I}, and w^{I}, the magnitudes of the corresponding stresses N_s^{I}, N_θ^{I}, and $N_{\theta s}^{\mathrm{I}}$ obtained by use of relations (2) are also of the order of $1/\rho^0$ and the moments are of $1/\rho^3$ and higher. The orders of magnitude of the stresses due to u^{II}, v^{II}, and w^{II} are not quite obvious and will be examined as follows.

Changing the variable s to y according to (7) and then to η such that

$$y = \eta^{1/\rho} \tag{27}$$

and using the displacements

$$\begin{aligned} u = u^{\mathrm{II}} &= \frac{1}{\rho^2}U \\ v = v^{\mathrm{II}} &= \frac{1}{\rho}V \\ w = w^{\mathrm{II}} &= W \end{aligned} \tag{28}$$

the elastic law (2), when the terms of lowest order of $1/\rho$ only are retained, leads to the following expressions

$$\begin{aligned} N_s^{\mathrm{II}} &= \frac{E\delta}{1-\nu^2}\left[\frac{1}{2}\eta\frac{\partial V}{\partial \eta} + \nu W \tan\alpha\right] \\ N_\theta^{\mathrm{II}} &= \frac{E\delta}{1-\nu^2}\left[W\tan\alpha + \frac{1}{2}\nu\eta\frac{\partial V}{\partial \eta}\right] \\ N_{s\theta}^{\mathrm{II}} = N_{\theta s}^{\mathrm{II}} &= \frac{E\delta}{1+\nu}\,\frac{1}{2}\,\frac{1}{\rho}\left[\frac{1}{2}\eta\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \theta}\sec\alpha\right] \\ M_s^{\mathrm{II}} &= E\delta L\tan^2\alpha\frac{1}{\rho^2}\left[\eta^2\frac{\partial^2 w}{\partial \eta^2} + \eta\frac{\partial w}{\partial \eta}\right] \\ M_\theta^{\mathrm{II}} &= \nu M_s^{\mathrm{II}} \\ M_{s\theta}^{\mathrm{II}} = M_{\theta s}^{\mathrm{II}} &= \frac{2E\delta}{1+\nu}L\tan^2\alpha\frac{1}{\rho^3}\eta\frac{\partial^2 w}{\partial \eta\,\partial \theta}\sec\alpha \end{aligned} \tag{29}$$

in which the relation

$$k = \frac{4}{\rho^4}(1-\nu^2)\tan^2\alpha \tag{30}$$

obtained from the expression (23) has been used.

† It is assumed as usual that the parameter m is limited to small values such that the differentiation with respect to the θ does not affect the order of magnitude. A study for large numbers of n was given by Steele [4].

Note that the normal stresses N_s^{II} and N_θ^{II} are of the same order as that of N_s^{I} and N_θ^{I}. It can be shown, however, that N_θ^{I} and N_s^{II} vanish identically.

Combining the two sets of solutions and when only the terms of the lowest order of $1/\rho$ are retained, one has

$$\begin{aligned} u &= u^{\text{I}}, & v &= v^{\text{I}}, & w &= w^{\text{I}} + w^{\text{II}} \\ N_s &= N_s^{\text{I}}, & N_\theta &= N_\theta^{\text{II}}, & N_{s\theta} &= N_{\theta s} = N_{s\theta}^{\text{I}} \\ M_s &= M_s^{\text{II}}, & M_\theta &= M_\theta^{\text{II}}, & M_{s\theta} &= M_{\theta s} = M_{s\theta}^{\text{II}}. \end{aligned} \tag{31}$$

Similarly, the transverse and tangential shearing forces defined by equations (10) and (20) are

$$S_\theta = S_\theta^{\text{II}}, \qquad S_s = S_s^{\text{II}}, \qquad T_s = T_s^{\text{I}} = N_{s\theta}^{\text{I}}. \tag{32}$$

Thus the two parts of the solution, membrane and bending, are coupled by the lateral deflection w; otherwise, they would be separable.

In view of equations (31), (32), and (28), and when solutions (24) and (25) are used, the stresses and moments may be given in the following final explicit forms:

$$\begin{aligned} N_s &= -2E\delta \tan\alpha \left[\frac{C_2}{m^2 - 2(1-v^2)} y^{-2} + \frac{3C_4}{m^2 - 7 + 2v} y^{-4}\right] \frac{\cos n\pi\theta}{\sin\ \theta_1} \\ N_\theta &= E\delta y^{-1} \tan\alpha \{ y^\rho [C_5 \cos(\rho \ln y) + C_6 \sin(\rho \ln y)] \\ &\quad + y^{-\rho}[C_7 \cos(\rho \ln y) + C_8 \sin(\rho \ln y)]\} \frac{\cos n\pi\theta}{\sin\ \theta_1} \\ N_{s\theta} &= T_s = \mp E\delta \left\{ \frac{6\tan\alpha}{m(m^2 - 7 + 2v)} C_4 y^{-4} \right\} \frac{\sin\ n\pi\theta}{\cos\ \theta_1} \\ M_s &= \frac{2E\delta}{\rho^2} \tan^2\alpha\, L y \{ y^\rho [C_6 \cos(\rho \ln y) - C_5 \sin(\rho \ln y)] \\ &\quad + y^{-\rho}[-C_8 \cos(\rho \ln y) + C_7 \sin(\rho \ln y)]\} \frac{\cos n\pi\theta}{\sin\ \theta_1} \\ M_\theta &= v M_s \\ S_\theta &= \mp \frac{2E\delta}{\rho^2} m(2-v) \tan^2\alpha\, y^{-1} \{ y^\rho [C_6 \cos(\rho \ln y) - C_5 \sin(\rho \ln y)] \\ &\quad + y^{-\rho}[-C_8 \cos(\rho \ln y) + C_7 \sin(\rho \ln y)]\} \frac{\sin\ n\pi\theta}{\cos\ \theta_1} \\ S_s &= \frac{E\delta}{\rho} \tan^2\alpha\, y^{-1} \{ y^\rho [(-C_5 + C_6) \cos(\rho \ln y) - (C_5 + C_6) \sin(\rho \ln y)] \\ &\quad + y^{-\rho}[(C_7 + C_8 \cos(\rho \ln y) - (C_7 + C_8) \sin(\rho \ln y)]\} \frac{\cos n\pi\theta}{\sin\ \theta_1} \end{aligned} \tag{33}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w^{\mathrm{II}}}{\partial s}$$

$$= \frac{1}{2L}\rho y^{-3}\{y^{\rho}[(C_5+C_6)\cos(\rho \ln y)-(C_5-C_6)\sin(\rho \ln y)]$$

$$-y^{-\rho}[(C_7-C_8)\cos(\rho \ln y)+(C_7+C_8)\sin(\rho \ln y)]\}\frac{\cos n\pi\theta}{\sin\ \theta_1}.$$

PARTICULAR SOLUTIONS DUE TO LATERAL NORMAL LOADS

Let the lateral normal load given by (11) be confined to the form

$$p_{rn}(y) = p_n L^{\beta} y^{2\beta} \tag{34}$$

where p_n and β are prescribed constants.

One may assume a set of particular solutions in the similar form as given by expressions (8) except λ_n. In this case λ_n shall be replaced by

$$\lambda^* = 2\beta+3 \tag{35}$$

a known number. By solving the three algebraic equations (12) simultaneously, the particular solutions are then readily obtained, provided that λ^* is not one of the roots of the determinant. However, when the load is uniformly distributed along meridians, $\beta = 0$ and $\lambda^* = 3$, which is one of the roots at the asymptotic case. In such cases, the approach needs to be modified. Since this is one of the most common loadings, the particular solution for this kind of uniform load will be given.

Because in this case λ^* is a finite constant as the parameter k approaches zero, the corresponding particular solution may be obtained from the equations of membrane theory of the system.

Setting $k = 0$ and having the independent variable s transformed to y, equations (6) reduce to the following equations of equilibrium of membrane theory:

$$\frac{1-\nu}{8}\left[y^2\frac{\partial^2 u}{\partial y^2}+3y\frac{\partial u}{\partial y}-8u\right]+\frac{1+\nu}{4}y\frac{\partial^2 u}{\partial y\,\partial\theta}\sec\alpha+\frac{\partial^2 u}{\partial\theta^2}\sec^2\alpha+(2-\nu)\frac{\partial v}{\partial\theta}\sec\alpha$$
$$+\frac{\partial w}{\partial\theta}\sec\alpha\tan\alpha = 0$$

$$\frac{1+\nu}{8}y\frac{\partial^2 u}{\partial y\,\partial\theta}\sec\alpha-\frac{3}{2}(1-\nu)\frac{\partial u}{\partial\theta}\sec\alpha+\frac{1}{4}y^2\frac{\partial^2 v}{\partial y^2}+\frac{3}{4}y\frac{\partial v}{\partial y} \tag{36}$$
$$+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial\theta^2}\sec^2\alpha-(1-\nu)v+\frac{1}{2}\nu y\frac{\partial w}{\partial y}\tan\alpha-(1-\nu)w\tan\alpha = 0$$
$$\frac{\partial u}{\partial\theta}\sec\alpha+\frac{1}{2}\nu y\frac{\partial v}{\partial y}+v+w\tan\alpha = \frac{1-\nu^2}{E\delta}Lp_r y^2$$

where

$$p_r = p_n \frac{\cos n\pi\theta}{\sin \theta_1}. \tag{37}$$

Let the particular solutions of equations (36) be assumed as below:

$$\begin{aligned} u^P &= \mp(d_1 + d_2 \ln y)y^2 \frac{\sin n\pi\theta}{\cos \theta_1} \\ v^P &= (b_1 + b_2 \ln y)y^2 \frac{\cos n\pi\theta}{\sin \theta_1} \\ w^P &= e_1(1 + \ln y)y^2 \frac{\cos n\pi\theta}{\sin \theta_1} \end{aligned} \tag{38}$$

in which d_1, d_2, b_1, b_2, and e_1 are constants to be determined. When these assumed solutions are substituted into equations (36) and after the sinusoidal functions and y^2 are cancelled, one will have three equations

$$J_l \ln y + H_l = \frac{Lp_n}{\tan\alpha} \frac{1-\nu^2}{E\delta} \delta_{lr} \tag{39}$$

where J_l and H_l are expressions of the physical and to-be-determined constants.

By making the coefficients of both sides of equations (39) equal, there are two sets of algebraic equations, each containing three equations:

$$J_l = 0 \tag{40}$$

and

$$H_l = \frac{Lp_n}{\tan\alpha} \frac{1-\nu^2}{E\delta} \delta_{lr}. \tag{41}$$

Only two of equations (40) are independent because $\lambda^* = 3$ is one of the roots of the determinant. Thus, the five constants may be determined by the five independent equations of (40) and (41). This results in

$$\begin{aligned} u^P &= \mp \frac{p_n}{\tan\alpha} \frac{L}{E\delta} \frac{m}{3} \left\{ \frac{1}{2m^2} [2m^4 - 3(5-\nu)m^2 - 3(1+\nu) + (m^2 - 7 + 2\nu) \ln y \right\} y^2 \frac{\sin n\pi\theta}{\cos \theta_1} \\ v^P &= \frac{p_n}{\tan\alpha} \frac{L}{E\delta} \frac{1}{6} [3(1-2\nu) - m^2] y^2 \frac{\cos n\pi\theta}{\sin \theta_1} \\ w^P &= \frac{p_n}{\tan^2\alpha} \frac{L}{E\delta} \frac{1}{3} m^2 [m^2 - 7 + 2\nu](1 + \ln y) \frac{\cos n\pi\theta}{\sin \theta_1}. \end{aligned} \tag{42}$$

When these displacements are substituted into the expressions (2) with $k = 0$, the corresponding stresses are

$$\begin{aligned} N_s^P &= \frac{p_n L}{\tan\alpha}\frac{1}{6}(3-m^2)y^2\frac{\cos n\pi\theta}{\sin\ \theta_1} \\ N_\theta^P &= \frac{p_n L}{\tan\alpha}y^2\frac{\cos n\pi\theta}{\sin\ \theta_1} \\ N_{s\theta}^P &= \pm\frac{p_n L}{\tan\alpha}\frac{m}{3}y^2\frac{\sin\ n\pi\theta}{\cos\ \theta_1}. \end{aligned} \tag{43}$$

These particular solutions combined with those given by solutions (24), (25), and (33) constitute the complete solutions.

AN EXAMPLE

For purpose of illustration, take a truncated semicircular cone with the two generators simply supported. Thus the lower set of the solutions are to be used. Let it be clamped at the smaller end at $s = L_1$ and free at the other end where $s = L$ so that

$$\begin{aligned} u = v = w = \frac{\partial w}{\partial s} = 0 \qquad & \text{at } y = \sqrt{\left(\frac{L_1}{L}\right)} \\ N_s = T_s = M_s = S_s = 0 \qquad & \text{at } y = 1. \end{aligned} \tag{44}$$

By making use of the first two in each of the foregoing two sets of boundary conditions, constants C_1, C_2, C_3, and C_4 can be determined; then the other four constants can be determined by the remaining four boundary conditions.

For numerical computations, the following values are assumed

$$\alpha = 75^\circ, \qquad \nu = \tfrac{1}{3}, \qquad \sqrt{\left(\frac{L_1}{L}\right)} = 0{\cdot}90. \tag{45}$$

Considering t/R as a parameter where R is the principal radius at a section of thickness t, one obtains $\delta = (t/R)\cos\alpha$. The eight roots of λ computed from expressions (16), (17), and expressions (21), (22) for asymptotic values for $n = 1$ and 2 are listed in Table 1.

TABLE 1. THE VALUES OF λ

λ	$\frac{t}{R}$	$n = 1$	$n = 2$	Asymptotic values
$\lambda_{1,2}$	0·004	$\pm 0{\cdot}999999$	$\pm 1{\cdot}0523$	± 1
	0·006	$\pm 0{\cdot}999997$	$\pm 1{\cdot}1142$	± 1
	0·008	$\pm 0{\cdot}999995$	$\pm 1{\cdot}1955$	$+ 1$
$\lambda_{3,4}$	0·004	$\pm 3{\cdot}00003$	$\pm 2{\cdot}9851$	± 3
	0·006	$\pm 3{\cdot}00007$	$\pm 2{\cdot}9663$	± 3
	0·008	$+ 3{\cdot}00013$	$\pm 2{\cdot}9397$	± 3
$\lambda_{57,68}$	0·004	$\pm 153{\cdot}27(1{\cdot}0027 \pm i)$	$\pm 152{\cdot}75(1{\cdot}0099 \pm i)$	$\pm 153{\cdot}48(1 \pm i)$
	0·006	$\pm 125{\cdot}09(1{\cdot}0035 \pm i)$	$\pm 124{\cdot}51(1{\cdot}0149 \pm i)$	$\pm 125{\cdot}32(1 + i)$
	0·008	$\pm 108{\cdot}28(1{\cdot}0045 \pm i)$	$\pm 107{\cdot}77(1{\cdot}0198 \pm i)$	$\pm 108{\cdot}53(1 \pm i)$

The asymptotic solutions of displacements, stresses, and moments computed may be given in the form:

$$G_n(y, \theta) = g_n(y) \frac{\sin n\pi\theta}{\cos \theta_1} \qquad n = 1 \text{ and } 2 \tag{46}$$

in which the function $g_n(y)$ are presented in Figs. 1–7.

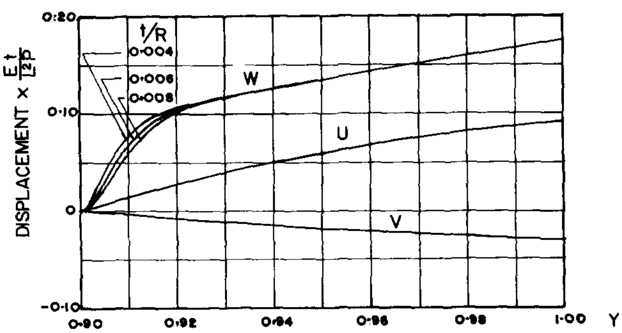

FIG. 1. Displacements u, v and w (n = 1).

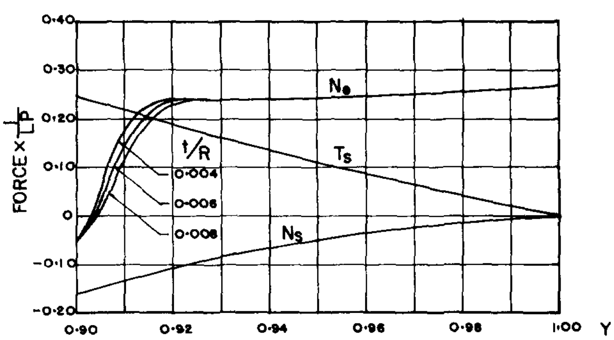

FIG. 2. Membrane forces N_θ, T_s and N_s (n = 1).

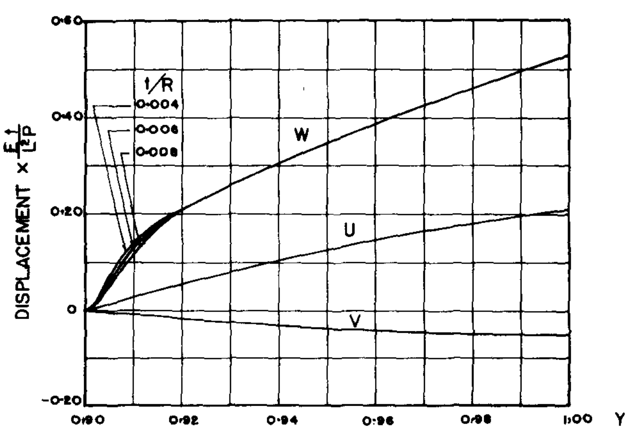

FIG. 3. Displacements u, v and w (n = 2).

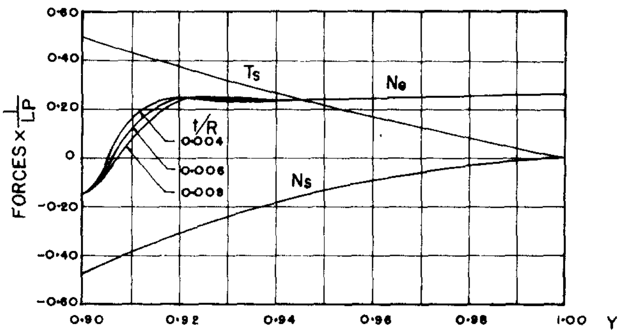

FIG. 4. Membrane forces N_θ, T_s and N_s ($n = 2$).

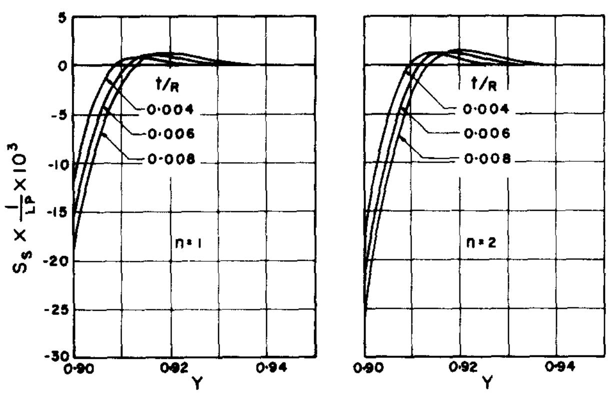

FIG. 5. Transverse shearing force S_s.

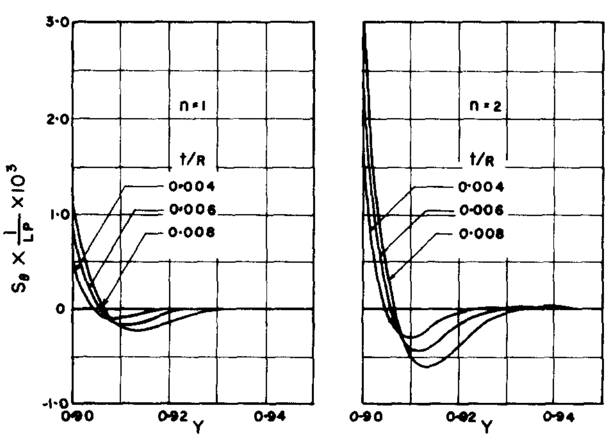

FIG. 6. Transverse shearing force S_θ.

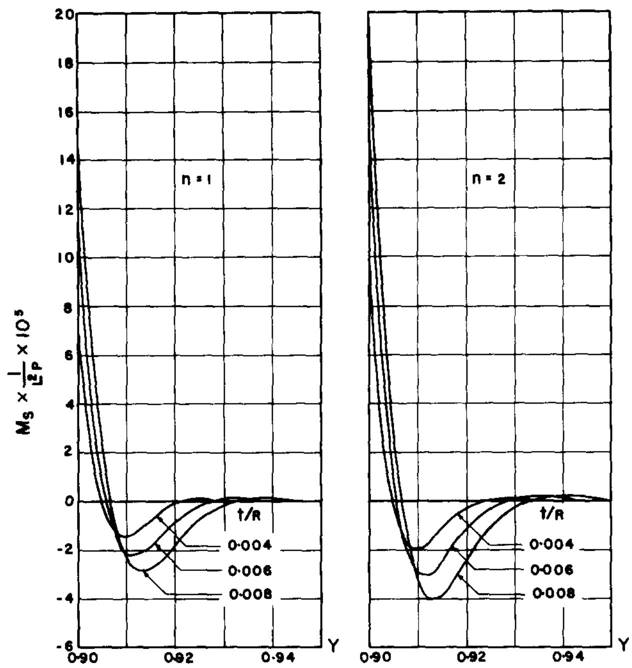

FIG. 7. Normal moment M_s.

When $n = 1$ the solution represents the response to a symmetrical lateral load. By a proper combination of solutions of $n = 1$ and 2, one may have the response to an asymmetrical load as shown in Fig. 8.

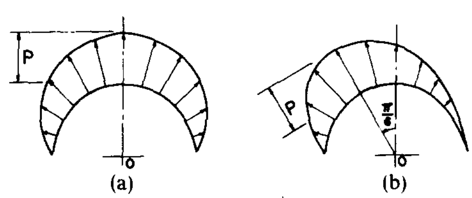

FIG. 8. (a) The symmetrical load. (b) The asymmetrical load.

CLOSING REMARKS

A number of approaches for solutions of shells of revolutions are available. A recent one was presented by Kalnins [5] by treating the system of equations as a series of initial-value problems. The method of asymptotic integration of differential equations with a large parameter has long been used in boundary layer theory in fluid mechanics. It has also been widely employed recently in the study of thin plates [6] and shells [2, 7].

The present asymptotic solutions are exact and applicable to conical shells when

$$\left[\frac{1}{12}\left(\frac{t}{R}\cos\alpha\right)^2\right]^{\frac{1}{4}} \ll 1.$$

When the above parameter is very small (as were those given in the example) the present explicit solutions will provide a good approximation for conical shells of constant thickness.

The following facts are not new but should be noted with interest. The moments and shearing forces due to the bending effect are of higher order than those of membrane stresses. However, the membrane stress N_θ induced by the bending effect is of the same order as the other membrane stresses. Thus, solutions of the membrane theory alone not only make the solutions incompatible with kinematic boundary conditions, but also introduce some non-negligible errors in the membrane stress N_θ.

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Résumé—La solution de couches coniques d'une épaisseur linéaire variable est épinglée sur une équation caractéristique de huitième degré. Dans ce rapport, une méthode pour résoudre cette équation est présentée, et résulte en une solution générale asymptotique. La solution particulière due à une charge latérale normale constante aux méridiens et ayant une distribution sinusoidale dans la direction circonférentielle, est donnée; est également inclu un exemple illustratif d'un cone semi-circulaire qui est simplement supporté le long de deux générateurs et ayant une extrémité fixée et l'autre libre.

Zusammenfassung—Die Lösung elastische Kegelschalen mit linear variabler Dicke hängt ab von einer charakteristischen Gleichung achten Grades. In der Arbeit wird eine Methode zur Lösung dieser Gleichung gegeben; das Resultat ist eine allgemein asymptotische Lösung. Eine Lösung des besonderen Falles einer seitlichen Normallast die den Meridianen entlang konstant ist und in der Umfangsrichtung eine sinusförmige Verteilung hat wird gegeben. Ferner wird als Illustration ein Beispiel eines halbrunden Kegels gegeben der an zwei Generatoren entlang gestützt wird, wobei das eine Ende fest ist und das andere frei.

Абстракт—Решение эластических конических оболочек линейно разнообразной толщины зависит от характеристического уравнения восьмого порядка. В этой статье даётся метод решения этого уравнения, в результате которого получается общее асимптотическое решение. Предлагается особое решение, обусловленное боковой нормальной нагрузкой, которая постоянна вдоль меридианов и имеет синусоидальное распространение в круговом направлении. Включён также пояснительный пример полукруглого конуса, который просто поддерживается вдоль двух генераторов с одним закреплённым и другим свободным концом.